On the energy spectrum of the damped quantum oscillator



1989 J. Phys. A: Math. Gen. 22 L361

(http://iopscience.iop.org/0305-4470/22/9/002)

J. Phys. A: Math. Gen. 22 (1989) L361-L363. Printed in the UK

LETTER TO THE EDITOR

# On the energy spectrum of the damped quantum oscillator

M S Wartak†

Department of Physics, University of California, Davis, CA 95616, USA

Received 18 November 1988, in final form 27 February 1989

**Abstract.** Feynman-Vernon theory is used to calculate the energy levels of a quantum harmonic oscillator interacting linearly with a heat bath.

One of the long-standing problems in theoretical physics appears to be the quantisation of dissipative systems [1, 2]. During the last few decades several approaches have been suggested. For example, Kanai [3] has introduced explicitly time-dependent Hamiltonian, solutions of which, however, seem to violate the uncertainty principle (see, e.g., the critiques by Dekker [4] and Greenberger [5]).

In the present letter we shall use the Feynman-Vernon theory [6] to calculate the energy spectrum (for large $n$) for a quantum harmonic oscillator interacting linearly with the reservoir which is chosen as a system of $N$ non-interacting harmonic oscillators. The Lagrangian in our problem is

$$L = L_S + L_B + L_{\text{coupling}} \tag{1}$$

and

$$L_S = \tfrac{1}{2}M\dot{x}^2 - \tfrac{1}{2}M\omega^2x^2$$

$$L_B = \sum_{i=1}^{N} (\tfrac{1}{2}m\dot{R}_i^2 - \tfrac{1}{2}m\omega_i^2R_i^2)$$

$$L_{\text{coupling}} = -x(t)\sum_{i=1}^{N} c_iR_i.$$

In the above formulae $x$ is the coordinate under consideration, $R_i$ is the coordinate of the $i$th particle of the bath which is coupled to the $x$ and $c_i$ is the corresponding coupling constant.

In order to evaluate the energy levels of our system we introduce the reduced Feynman propagator defined as [7]

$$\tilde{K}(x, \beta\hbar; x', 0) = \int \mathrm{d}\boldsymbol{R}\, K(x, \boldsymbol{R}, \beta\hbar; x', \boldsymbol{R}', 0)|_{\boldsymbol{R}=\boldsymbol{R}'} \tag{2}$$

where

$$K(x, \boldsymbol{R}, \beta\hbar; x', \boldsymbol{R}', 0) = \int_{x(0)=x'}^{x(\beta\hbar)=x} \int_{\boldsymbol{R}(0)=\boldsymbol{R}'}^{\boldsymbol{R}(\beta\hbar)=\boldsymbol{R}} \mathrm{D}x\, \mathrm{D}\boldsymbol{R} \exp\frac{\mathrm{i}}{\hbar}S[x, \boldsymbol{R}] \tag{3}$$

and $S[x, \boldsymbol{R}] = \int_0^\beta \mathrm{d}t\, L(t)$ which is the action of the total system.

† Present address: National Research Council, Department of Electrical Engineering, Ottawa, Ontario K1A 0R8, Canada.

 

The reduced propagator $\tilde{K}(x, \beta\hbar; x', 0)$ can be written as

$$\tilde{K}(x, \beta\hbar; x', 0) = \int \mathrm{D}x\, F[x] \exp\frac{\mathrm{i}}{\hbar} S_s[x] \tag{4}$$

where the influence functional $F[x]$ is by definition

$$F[x] = \int \mathrm{d}\boldsymbol{R} \int \mathrm{D}\boldsymbol{R} \exp\frac{\mathrm{i}}{\hbar}(S_{\mathrm{B}} + S_{\mathrm{coupling}}). \tag{5}$$

For Lagrangian (1) all the integrals appearing in $F[x]$ are Gaussian and can be easily calculated. The result is

$$F[x] = F_0(\beta\hbar) \exp\frac{\mathrm{i}}{\hbar} \int_0^{\beta\hbar} \mathrm{d}t \int_0^{\beta\hbar} \mathrm{d}t'\, x(t)x(t')\alpha(t-t') \tag{6}$$

and $F_0(\beta\hbar) = \exp - \Sigma_{k=1}^{N} \ln(2\mathrm{i} \sin \frac{1}{2}\beta\hbar\omega_k)$. In the classical limit $\beta\hbar \to 0$ $(\beta \equiv 1/k_{\mathrm{B}}T)$, the kernel $\alpha(t-t')$ is

$$\alpha(t-t') = \frac{c^2\rho_0\pi}{2m} \delta'(|t-t'|) \tag{7}$$

where the constant $\rho_0$ characterises density of states of the reservoir.

In order to calculate $\tilde{K}(x, \beta\hbar; x', 0)$, we first expand the action $S[x]$ around the stationary point $x_{\mathrm{c}}(t)$ which is given by

$$M\ddot{x}_{\mathrm{c}}(t) + M\omega^2 x_{\mathrm{c}}(t) = 2\int_0^{\beta\hbar} \mathrm{d}t'\, x_{\mathrm{c}}(t')\alpha(t-t') \tag{8}$$

along with the conditions $x_{\mathrm{c}}(0) = x_{\mathrm{c}}(\beta\hbar) = x$. Using (7), equation (8) reduces to

$$\ddot{x}_{\mathrm{c}}(t) + \omega^2 x_{\mathrm{c}}(t) + \gamma\dot{x}_{\mathrm{c}}(t) = 0 \tag{9}$$

with $\gamma = c^2\rho_0\pi/(2mM)$. Using the solution of (9) one finds

$$S[x_{\mathrm{c}}] = \tfrac{1}{2}Mx^2\lambda\left(\cot\tfrac{1}{2}\lambda\beta\hbar - \frac{\cosh\frac{1}{2}\gamma\beta\hbar}{\sin\frac{1}{2}\lambda\beta\hbar}\right) \tag{10}$$

where $\lambda^2 = 4\omega^2 - \gamma^2 > 0$. The reduced propagator can be calculated by using a method described by Coleman [8] with the result:

$$\tilde{K}(x, \beta\hbar; x, 0) = \left(\frac{M}{2\pi\mathrm{i}\hbar\beta\hbar}\right)^{1/2} \left(\frac{\det D_0}{\det D}\right)^{1/2} \exp\frac{\mathrm{i}}{\hbar} S[x_{\mathrm{c}}] \tag{11}$$

where $D_0$ corresponds to a free particle, i.e. $D_0 \equiv -M\,\mathrm{d}^2/\mathrm{d}t^2$, and

$$\int_0^{\beta\hbar} D(t-t')\eta_n(t')\,\mathrm{d}t' = \varepsilon_n\eta_n(t) \tag{12}$$

where

$$D(t-t') = \delta(t-t')\left(-M\frac{\mathrm{d}^2}{\mathrm{d}t^2} - M\omega^2\right) + 2\alpha(t-t'). \tag{13}$$

Again, using (7) one can reduce (12) into the following equation:

$$\left(M\frac{\mathrm{d}^2}{\mathrm{d}t^2} + M\omega^2\right)\eta_n(t) + \gamma M\dot{\eta}_n(t) = -\varepsilon_n\eta_n(t) \tag{14}$$

and $\eta_n(0)=\eta_n(\beta\hbar)=0$. Using the solution of (14) one finds

$$\frac{\det D_0}{\det D}=\frac{\sqrt{(\omega\beta\hbar)^2-(\frac{1}{2}\gamma\beta\hbar)^2}}{\sin\sqrt{(\omega\rho\hbar)^2-(\frac{1}{2}\gamma\beta\hbar)^2}}. \tag{15}$$

Combining (10), (11) and (15) we can write the propagator as

$$\tilde{K}(x,\beta\hbar;x,0)=\left(\frac{M\lambda}{4\pi \mathrm{i}\hbar\sin\frac{1}{2}\lambda\beta\hbar}\right)^{1/2}\exp\left(\frac{\mathrm{i}M\lambda x^2}{2\hbar\sin\frac{1}{2}\lambda\beta\hbar}(\cos\tfrac{1}{2}\lambda\beta\hbar-\cosh\tfrac{1}{2}\gamma\beta\hbar)\right) \tag{16}$$

which is reduced to the Feynman and Hibbs result [9] for $\gamma=0$ (free particle case).

The energy levels of the oscillator can be calculated by the Feynman method [9]. For large $n$, when $\frac{1}{2}$ can be neglected with respect to $n$, one finds easily

$$E_n\simeq\hbar n\sqrt{4\omega^2-\gamma^2}\simeq n\omega\hbar\left[1-\frac{1}{2}\left(\frac{\gamma}{2\omega}\right)^2\right] \tag{17}$$

where in the last step we have used the fact that $\gamma$ is small. Equation (17) tells us that for small linear coupling to the reservoir, the energy levels of the harmonic oscillator, for large $n$, are reduced by a factor of $\frac{1}{2}(\gamma/2\omega)^2$.

Let us compare our result with others. In the canonical approach used by Tartaglia [10], the expectation value of the Hamiltonian, for the damped quantum oscillator, is

$$\langle\hat{H}\rangle_n=(n+\tfrac{1}{2})\left(\hbar\omega+\frac{\hbar\gamma^2}{4\omega}\right) \tag{18}$$

which is time independent, but the expectation value of the energy $\langle E\rangle_n=\langle H\rangle_n\exp(-\gamma t)$ is not. Similar modification was found by Hasse [11]. His expectation value of the Hamiltonian is

$$\langle H\rangle_n=(n+\tfrac{1}{2})\frac{\hbar\omega^2}{\Omega} \tag{19}$$

where $\Omega=(\omega^2-\frac{1}{4}\gamma^2)^{1/2}$ is the frequency reduced by damping. It stays constant in time but the expectation value of energy $\langle E\rangle_n=\langle H\rangle_n\exp(-\gamma t)$ does not.

The author would like to thank the referee for useful remarks.

## References


[1] Messer J 1979 *Acta Phys. Austriaca* **50** 75
[2] Dekker H 1981 *Phys. Rep.* **80** 1
[3] Kanai E 1948 *Prog. Theor. Phys.* **3** 440
[4] Dekker H 1977 *Phys. Rev.* A **16** 2126
[5] Greenberger D M 1979 *J. Math. Phys.* **20** 762
[6] Feynman R P and Vernon F L 1963 *Ann. Phys., NY* **24** 118
[7] Caldeira A O and Leggett A J 1981 *Phys. Rev. Lett.* **46** 211; 1983 *Ann. Phys., NY* **149** 374
[8] Coleman S 1977 *Phys. Rev.* D **15** 2929
Callan C G and Coleman S 1977 *Phys. Rev.* D **16** 1762
[9] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals*
[10] Tartaglia A 1977 *Lett. Nuovo Cimento* **19** 205
[11] Hasse R W 1975 *J. Math. Phys.* **16** 2005